ON SOME NEW HERMITE-HADAMARD TYPE INEQUALITIES FOR $s-$GEOMETRICALLY CONVEX FUNCTIONS

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Abstract. In this paper, some new integral inequalities of Hermite-Hadamard type related to the $s-$geometrically convex functions are established and some applications to special means of positive real numbers are also given.

1. Introduction

In this section, we firstly list several definitions and some known results.

Definition 1. *Let I be an interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}$ is said to be convex if*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Recently, In [2], the concept of geometrically and $s-$geometrically convex functions was introduced as follows:

Definition 2. *A function $f : I \subset \mathbb{R}_+ = (0, \infty) \to \mathbb{R}_+$ is said to be a geometrically convex function if*

$$f\left(x^{\lambda} y^{1-\lambda}\right) \leq f(x)^{\lambda} f(y)^{1-\lambda}$$

for $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 3. *A function $f : I \subset \mathbb{R}_+ = (0, \infty) \to \mathbb{R}_+$ is said to be a geometrically convex function if*

$$f\left(x^{\lambda} y^{1-\lambda}\right) \leq f(x)^{\lambda^s} f(y)^{(1-\lambda)^s}$$

for some $s \in (0, 1]$, where $x, y \in I$ and $\lambda \in [0, 1]$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality is well known in the literature as Hermite-Hadamard integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In [2], The authors has established some integral inequalities connected with the inequalities (1.1) for the $s-$geometrically convex and monotonically decreasing functions. In [1], Tunc has established inequalities for $s-$geometrically and geometrically convex functions which are connected with the famous Hermite Hadamard inequality holding for convex functions. In [1] also Tunc has given the following result for geometrically convex and monotonically decreasing functions:

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Corollary 1. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be geometrically convex and monotonically decreasing on $[a,b]$, then one has*

$$f^2\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} f\left(\frac{ab}{x}\right) dx \leq f(a)f(b). \tag{1.2}$$

We note that, the inequalities (1.2) are also true without the condition monotonically decreasing and the inequalities (1.2) are sharp.

In this paper, the author give new identity for differentiable functions. A consequence of the identity is that the author establish some new inequalities connected with the inequalities (1.2) for the $s-$geometrically convex functions.

2. Main Results

In order to prove our results, we need the following lemma:

Lemma 1. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on I°, and $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then*

$$\begin{aligned} & f(a)f(b) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} f\left(\frac{ab}{x}\right) dx \\ = & \int_0^1 \frac{b}{2}\ln\left(\frac{a}{b}\right)(t-1)\left(\frac{a}{b}\right)^{\frac{t}{2}} f\left(a^{1-t}(ab)^{\frac{t}{2}}\right) f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right) \\ & + \frac{a}{2}\ln\left(\frac{b}{a}\right)(t-1)\left(\frac{b}{a}\right)^{\frac{t}{2}} f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right) f\left(b^{1-t}(ab)^{\frac{t}{2}}\right) dt, \end{aligned} \tag{2.1}$$

$$\begin{aligned} & f^2\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} f\left(\frac{ab}{x}\right) dx \\ = & \int_0^1 \frac{b}{2}\ln\left(\frac{a}{b}\right) t \left(\frac{a}{b}\right)^{\frac{t}{2}} f\left(a^{1-t}(ab)^{\frac{t}{2}}\right) f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right) \\ & + \frac{a}{2}\ln\left(\frac{b}{a}\right) t \left(\frac{b}{a}\right)^{\frac{t}{2}} f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right) f\left(b^{1-t}(ab)^{\frac{t}{2}}\right) dt. \end{aligned} \tag{2.2}$$

Proof. Integrating by part and changing variables of integration yields

$$\begin{aligned}
& \int_0^1 \frac{b}{2}\ln\left(\frac{a}{b}\right)(t-1)\left(\frac{a}{b}\right)^{\frac{t}{2}} f\left(a^{1-t}(ab)^{\frac{t}{2}}\right) f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right) \\
& +\frac{a}{2}\ln\left(\frac{b}{a}\right)(t-1)\left(\frac{b}{a}\right)^{\frac{t}{2}} f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right) f\left(b^{1-t}(ab)^{\frac{t}{2}}\right) dt \\
= & \int_0^1 (t-1)\left[f\left(a^{1-t}(ab)^{\frac{t}{2}}\right) f\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right]' dt \\
= & (t-1) f\left(a^{1-t}(ab)^{\frac{t}{2}}\right) f\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\Big|_0^1 - \int_0^1 f\left(a^{1-t}(ab)^{\frac{t}{2}}\right) f\left(b^{1-t}(ab)^{\frac{t}{2}}\right) dt \\
= & f(a)f(b) - \frac{2}{\ln b - \ln a}\int_a^{\sqrt{ab}} \frac{f(x)}{x} f\left(\frac{ab}{x}\right) dx.
\end{aligned}$$

By the following equality, we obtain the inequality (2.1)

$$\int_a^{\sqrt{ab}} \frac{f(x)}{x} f\left(\frac{ab}{x}\right) dx = \int_{\sqrt{ab}}^b \frac{f(x)}{x} f\left(\frac{ab}{x}\right) dx.$$

$$\begin{aligned}
& \int_0^1 \frac{b}{2}\ln\left(\frac{a}{b}\right) t\left(\frac{a}{b}\right)^{\frac{t}{2}} f\left(a^{1-t}(ab)^{\frac{t}{2}}\right) f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right) \\
& +\frac{a}{2}\ln\left(\frac{b}{a}\right) t\left(\frac{b}{a}\right)^{\frac{t}{2}} f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right) f\left(b^{1-t}(ab)^{\frac{t}{2}}\right) dt \\
= & \int_0^1 t\left[f\left(a^{1-t}(ab)^{\frac{t}{2}}\right) f\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right]' dt \\
= & tf\left(a^{1-t}(ab)^{\frac{t}{2}}\right) f\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\Big|_0^1 - \int_0^1 f\left(a^{1-t}(ab)^{\frac{t}{2}}\right) f\left(b^{1-t}(ab)^{\frac{t}{2}}\right) dt \\
= & f^2\left(\sqrt{ab}\right) - \frac{2}{\ln b - \ln a}\int_a^{\sqrt{ab}} \frac{f(x)}{x} f\left(\frac{ab}{x}\right) dx \\
= & f^2\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a}\int_a^{b} \frac{f(x)}{x} f\left(\frac{ab}{x}\right) dx.
\end{aligned}$$

This completes the proof of Lemma 1. □

Theorem 1. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, and $a, b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is $s-$geometrically convex on $[a,b]$ for $q \geq 1$ and $s \in (0,1]$,*

then

(2.3)

$$\left| f(a)f(b)-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}f\left(\frac{ab}{x}\right)dx\right| \leq \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}} H_1\left(s,q;h_1(\theta),h_1(\vartheta)\right),$$

(2.4)

$$\left| f^2\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}f\left(\frac{ab}{x}\right)dx\right| \leq \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}} H_2\left(s,q;h_2(\theta),h_2(\vartheta)\right),$$

where $M_1=\max\limits_{x\in\left[a,\sqrt{ab}\right]}|f(x)|,\ M_2=\max\limits_{x\in\left[\sqrt{ab},b\right]}|f(x)|,$

$$h_1(\theta)=\begin{cases}\frac{1}{2}, & \theta=1\\ \frac{\theta-\ln\theta-1}{(\ln\theta)^2}, & \theta\neq 1\end{cases},\quad h_2(\theta)=\begin{cases}\frac{1}{2}, & \theta=1\\ \frac{\theta\ln\theta-\theta+1}{(\ln\theta)^2}, & \theta\neq 1\end{cases}, \tag{2.5}$$

(2.6)

$$\theta\left(u,v\right)=a^{q/2}b^{-q/2}\left|f'(a)\right|^{u}\left|f'(b)\right|^{-v},\ \vartheta\left(u,v\right)=a^{-q/2}b^{q/2}\left|f'(a)\right|^{-u}\left|f'(b)\right|^{v},\ u,v>0,$$

(2.7) $H_i\left(s,q;h_i(\theta),h_i(\vartheta)\right)$

$$=\left\{\begin{array}{c} b\left|f'(b)\right|^{s}M_1h_i^{1/q}\left(\theta\left(\frac{qs}{2},\frac{qs}{2}\right)\right)+a\left|f'(a)\right|^{s}M_2h_i^{1/q}\left(\vartheta\left(\frac{qs}{2},\frac{qs}{2}\right)\right),\\ \left|f'(a)\right|,\ \left|f'(b)\right|\leq 1,\\ b\left|f'(b)\right|^{1/s}M_1h_i^{1/q}\left(\theta\left(\frac{q}{2s},\frac{q}{2s}\right)\right)+a\left|f'(a)\right|^{1/s}M_2h_i^{1/q}\left(\vartheta\left(\frac{q}{2s},\frac{q}{2s}\right)\right),\\ \left|f'(a)\right|,\ \left|f'(b)\right|\geq 1,\\ b\left|f'(b)\right|^{1/s}M_1h_i^{1/q}\left(\theta\left(\frac{qs}{2},\frac{q}{2s}\right)\right)+a\left|f'(a)\right|^{s}M_2h_i^{1/q}\left(\vartheta\left(\frac{qs}{2},\frac{q}{2s}\right)\right),\\ \left|f'(a)\right|\leq 1\leq\ \left|f'(b)\right|,\\ b\left|f'(b)\right|^{s}M_1h_i^{1/q}\left(\theta\left(\frac{q}{2s},\frac{qs}{2}\right)\right)+a\left|f'(a)\right|^{1/s}M_2h_i^{1/q}\left(\vartheta\left(\frac{q}{2s},\frac{qs}{2}\right)\right),\\ \left|f'(b)\right|\leq 1\leq\ \left|f'(a)\right|.\end{array}\right.,i=1,2$$

Proof. (1) Let $M_1=\max\limits_{x\in\left[a,\sqrt{ab}\right]}|f(x)|,\ M_2=\max\limits_{x\in\left[\sqrt{ab},b\right]}|f(x)|$. Since $|f'|^q$ is $s-$geometrically convex on $[a,b]$, from lemma 1 and Hölder inequality, we have

$$\left| f(a)f(b)-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}f\left(\frac{ab}{x}\right)dx\right|$$

$$\leq \int_0^1 \frac{b}{2}\left|\ln\left(\frac{a}{b}\right)\right||t-1|\left(\frac{a}{b}\right)^{\frac{t}{2}}\left|f\left(a^{1-t}\left(ab\right)^{\frac{t}{2}}\right)\right|\left|f'\left(b^{1-t}\left(ab\right)^{\frac{t}{2}}\right)\right|$$

$$+\frac{a}{2}\ln\left(\frac{b}{a}\right)|t-1|\left(\frac{b}{a}\right)^{\frac{t}{2}}\left|f'\left(a^{1-t}\left(ab\right)^{\frac{t}{2}}\right)\right|\left|f\left(b^{1-t}\left(ab\right)^{\frac{t}{2}}\right)\right|dt$$

$$\leq \frac{b}{2}\left|\ln\left(\frac{a}{b}\right)\right| M_1 \int_0^1 (1-t)\left(\frac{a}{b}\right)^{\frac{t}{2}} \left|f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right| dt$$

$$+\frac{a}{2}\ln\left(\frac{b}{a}\right) M_2 \int_0^1 (1-t)\left(\frac{b}{a}\right)^{\frac{t}{2}} \left|f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right| dt$$

$$\leq \frac{b}{2}\left|\ln\left(\frac{a}{b}\right)\right| M_1 \left(\int_0^1 (1-t)\,dt\right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)\left(\frac{a}{b}\right)^{\frac{qt}{2}} \left|f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{a}{2}\ln\left(\frac{b}{a}\right) M_2 \left(\int_0^1 (1-t)\,dt\right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)\left(\frac{b}{a}\right)^{\frac{qt}{2}} \left|f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{b}{2}\ln\left(\frac{b}{a}\right) M_1 \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)\left(\frac{a}{b}\right)^{\frac{qt}{2}} \left|f'(a)\right|^{q(t/2)^s} \left|f'(b)\right|^{q((2-t)/2)^s} dt\right)^{\frac{1}{q}}$$

$$+\frac{a}{2}\ln\left(\frac{b}{a}\right) M_2 \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)\left(\frac{b}{a}\right)^{\frac{qt}{2}} \left|f'(b)\right|^{q(t/2)^s} \left|f'(a)\right|^{q((2-t)/2)^s} dt\right)^{\frac{1}{q}}. \tag{2.8}$$

If $0<\mu\leq 1\leq \eta,\ 0<\alpha, s\leq 1$, then

$$\mu^{\alpha^s} \leq \mu^{\alpha s}, \quad \eta^{\alpha^s} \leq \eta^{\alpha/s}. \tag{2.9}$$

(i) If $1 \geq |f'(a)|,\ |f'(b)|$, by (2.9) we obtain that

$$\int_0^1 (1-t)\left(\frac{a}{b}\right)^{\frac{qt}{2}} \left|f'(a)\right|^{q(t/2)^s} \left|f'(b)\right|^{q((2-t)/2)^s} dt$$

$$\leq \int_0^1 (1-t)\left(\frac{a}{b}\right)^{\frac{qt}{2}} \left|f'(a)\right|^{\frac{qst}{2}} \left|f'(b)\right|^{q\frac{qs(2-t)}{2}} dt = \left|f'(b)\right|^{qs} h_1\left(\theta\left(\frac{qs}{2},\frac{qs}{2}\right)\right),$$

$$\int_0^1 (1-t)\left(\frac{b}{a}\right)^{\frac{qt}{2}} \left|f'(b)\right|^{q(t/2)^s} \left|f'(a)\right|^{q((2-t)/2)^s} dt \tag{2.10}$$

$$\leq \int_0^1 (1-t)\left(\frac{b}{a}\right)^{\frac{qt}{2}} \left|f'(b)\right|^{\frac{qst}{2}} \left|f'(a)\right|^{\frac{qs(2-t)}{2}} dt = \left|f'(a)\right|^{qs} h_1\left(\vartheta\left(\frac{qs}{2},\frac{qs}{2}\right)\right).$$

(ii) If $1 \leq |f'(a)|,\ |f'(b)|$, by (2.9) we obtain that

$$\int_0^1 (1-t)\left(\frac{a}{b}\right)^{\frac{qt}{2}} \left|f'(a)\right|^{q(t/2)^s} \left|f'(b)\right|^{q((2-t)/2)^s} dt$$

$$\leq \int_0^1 (1-t)\left(\frac{a}{b}\right)^{\frac{qt}{2}} \left|f'(a)\right|^{\frac{qt}{2s}} \left|f'(b)\right|^{\frac{q(2-t)}{2s}} dt = \left|f'(b)\right|^{q/s} h_1\left(\theta\left(\frac{q}{2s},\frac{q}{2s}\right)\right),$$

$$\begin{aligned}
&\int_0^1 (1-t)\left(\frac{b}{a}\right)^{\frac{qt}{2}} \left|f'(b)\right|^{q(t/2)^s} \left|f'(a)\right|^{q((2-t)/2)^s} dt \\
&\leq \int_0^1 (1-t)\left(\frac{b}{a}\right)^{\frac{qt}{2}} \left|f'(b)\right|^{\frac{qt}{2s}} \left|f'(a)\right|^{\frac{q(2-t)}{2s}} dt = \left|f'(a)\right|^{q/s} h_1\left(\vartheta\left(\frac{q}{2s},\frac{q}{2s}\right)\right).
\end{aligned} \tag{2.11}$$

(iii) If $|f'(a)| \leq 1 \leq |f'(b)|$, by (2.9) we obtain that

$$\begin{aligned}
&\int_0^1 (1-t)\left(\frac{a}{b}\right)^{\frac{qt}{2}} \left|f'(a)\right|^{q(t/2)^s} \left|f'(b)\right|^{q((2-t)/2)^s} dt \\
&\leq \int_0^1 (1-t)\left(\frac{a}{b}\right)^{\frac{qt}{2}} \left|f'(a)\right|^{\frac{qst}{2}} \left|f'(b)\right|^{\frac{q(2-t)}{2s}} dt = \left|f'(b)\right|^{q/s} h_1\left(\theta\left(\frac{qs}{2},\frac{q}{2s}\right)\right),
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 (1-t)\left(\frac{b}{a}\right)^{\frac{qt}{2}} \left|f'(b)\right|^{q(t/2)^s} \left|f'(a)\right|^{q((2-t)/2)^s} dt \\
&\leq \int_0^1 (1-t)\left(\frac{b}{a}\right)^{\frac{qt}{2}} \left|f'(b)\right|^{\frac{qt}{2s}} \left|f'(a)\right|^{\frac{qs(2-t)}{2}} dt = \left|f'(a)\right|^{qs} h_1\left(\vartheta\left(\frac{qs}{2},\frac{q}{2s}\right)\right).
\end{aligned} \tag{2.12}$$

(iv) If $|f'(b)| \leq 1 \leq |f'(a)|$, by (2.9) we obtain that

$$\begin{aligned}
&\int_0^1 (1-t)\left(\frac{a}{b}\right)^{\frac{qt}{2}} \left|f'(a)\right|^{q(t/2)^s} \left|f'(b)\right|^{q((2-t)/2)^s} dt \\
&\leq \int_0^1 (1-t)\left(\frac{a}{b}\right)^{\frac{qt}{2}} \left|f'(a)\right|^{\frac{qt}{2s}} \left|f'(b)\right|^{\frac{qs(2-t)}{2}} dt = \left|f'(b)\right|^{qs} h_1\left(\theta\left(\frac{q}{2s},\frac{qs}{2}\right)\right),
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 (1-t)\left(\frac{b}{a}\right)^{\frac{qt}{2}} \left|f'(b)\right|^{q(t/2)^s} \left|f'(a)\right|^{q((2-t)/2)^s} dt \\
&\leq \int_0^1 (1-t)\left(\frac{b}{a}\right)^{\frac{qt}{2}} \left|f'(b)\right|^{\frac{qst}{2}} \left|f'(a)\right|^{\frac{q(2-t)}{2s}} dt = \left|f'(a)\right|^{q/s} h_1\left(\vartheta\left(\frac{q}{2s},\frac{qs}{2}\right)\right).
\end{aligned} \tag{2.13}$$

From (2.8) to (2.13), (2.3) holds.

(2) Let $M_1 = \max_{x\in[a,\sqrt{ab}]} |f(x)|$, $M_2 = \max_{x\in[\sqrt{ab},b]} |f(x)|$. Since $|f'|^q$ is $s-$geometrically convex on $[a,b]$, from lemma 1 and Hölder inequality, we have

$$\begin{aligned}
&\left| f^2\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} f\left(\frac{ab}{x}\right) dx \right| \\
&\leq \int_0^1 \frac{b}{2}\left|\ln\left(\frac{a}{b}\right)\right| t\left(\frac{a}{b}\right)^{\frac{t}{2}} \left| f\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right| \left| f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right| \\
&\quad + \frac{a}{2}\ln\left(\frac{b}{a}\right) t\left(\frac{b}{a}\right)^{\frac{t}{2}} \left| f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right| \left| f\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right| dt \\
&\leq \frac{b}{2}\ln\left(\frac{b}{a}\right) M_1 \int_0^1 t\left(\frac{a}{b}\right)^{\frac{t}{2}} \left| f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right| dt \\
&\quad + \frac{a}{2}\ln\left(\frac{b}{a}\right) M_2 \int_0^1 t\left(\frac{b}{a}\right)^{\frac{t}{2}} \left| f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right| dt \\
&\leq \frac{b}{2}\ln\left(\frac{b}{a}\right) M_1 \left(\int_0^1 t dt\right)^{1-\frac{1}{q}} \left(\int_0^1 t\left(\frac{a}{b}\right)^{\frac{qt}{2}} \left| f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad + \frac{a}{2}\ln\left(\frac{b}{a}\right) M_2 \left(\int_0^1 t dt\right)^{1-\frac{1}{q}} \left(\int_0^1 t\left(\frac{b}{a}\right)^{\frac{qt}{2}} \left| f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{b}{2}\ln\left(\frac{b}{a}\right) M_1 \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\int_0^1 t\left(\frac{a}{b}\right)^{\frac{qt}{2}} |f'(a)|^{q(t/2)^s} |f'(b)|^{q((2-t)/2)^s} dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\frac{a}{2}\ln\left(\frac{b}{a}\right) M_2 \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\int_0^1 t\left(\frac{b}{a}\right)^{\frac{qt}{2}} |f'(b)|^{q(t/2)^s} |f'(a)|^{q((2-t)/2)^s} dt\right)^{\frac{1}{q}}. \tag{2.14}$$

(i) If $1 \geq |f'(a)|, |f'(b)|$, by (2.9) we obtain that

$$\int_0^1 t\left(\frac{a}{b}\right)^{\frac{qt}{2}} |f'(a)|^{q(t/2)^s} |f'(b)|^{q((2-t)/2)^s} dt = |f'(b)|^{qs} h_2\left(\theta\left(\frac{qs}{2},\frac{qs}{2}\right)\right),$$

$$\int_0^1 t\left(\frac{b}{a}\right)^{\frac{qt}{2}} |f'(b)|^{q(t/2)^s} |f'(a)|^{q((2-t)/2)^s} dt = |f'(a)|^{qs} h_2\left(\vartheta\left(\frac{qs}{2},\frac{qs}{2}\right)\right). \tag{2.15}$$

(ii) If $1 \leq |f'(a)|, |f'(b)|$, by (2.9) we obtain that

$$\int_0^1 t\left(\frac{a}{b}\right)^{\frac{qt}{2}} |f'(a)|^{q(t/2)^s} |f'(b)|^{q((2-t)/2)^s} dt \leq |f'(b)|^{q/s} h_2\left(\theta\left(\frac{q}{2s},\frac{q}{2s}\right)\right),$$

$$\int_0^1 t\left(\frac{b}{a}\right)^{\frac{qt}{2}} \left|f'(b)\right|^{q(t/2)^s} \left|f'(a)\right|^{q((2-t)/2)^s} dt \le \left|f'(a)\right|^{q/s} h_2\left(\vartheta\left(\frac{q}{2s},\frac{q}{2s}\right)\right). \tag{2.16}$$

(iii) If $|f'(a)| \le 1 \le |f'(b)|$, by (2.9) we obtain that

$$\int_0^1 t\left(\frac{a}{b}\right)^{\frac{qt}{2}} \left|f'(a)\right|^{q(t/2)^s} \left|f'(b)\right|^{q((2-t)/2)^s} dt \le \left|f'(b)\right|^{q/s} h_2\left(\theta\left(\frac{qs}{2},\frac{q}{2s}\right)\right),$$

$$\int_0^1 t\left(\frac{b}{a}\right)^{\frac{qt}{2}} \left|f'(b)\right|^{q(t/2)^s} \left|f'(a)\right|^{q((2-t)/2)^s} dt \le \left|f'(a)\right|^{qs} h_2\left(\vartheta\left(\frac{qs}{2},\frac{q}{2s}\right)\right). \tag{2.17}$$

(iv) If $|f'(b)| \le 1 \le |f'(a)|$, by (2.9) we obtain that

$$\int_0^1 t\left(\frac{a}{b}\right)^{\frac{qt}{2}} \left|f'(a)\right|^{q(t/2)^s} \left|f'(b)\right|^{q((2-t)/2)^s} dt \le \left|f'(b)\right|^{qs} h_2\left(\theta\left(\frac{q}{2s},\frac{qs}{2}\right)\right),$$

$$\int_0^1 t\left(\frac{b}{a}\right)^{\frac{qt}{2}} \left|f'(b)\right|^{q(t/2)^s} \left|f'(a)\right|^{q((2-t)/2)^s} dt \le \left|f'(a)\right|^{q/s} h_2\left(\vartheta\left(\frac{q}{2s},\frac{qs}{2}\right)\right). \tag{2.18}$$

From (2.14) to (2.18), (2.4) holds. This completes the required proof. □

If taking $s = 1$ in Theorem 1, we can derive the following corollary.

Corollary 2. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, and $a, b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is geometrically convex on $[a,b]$ for $q \ge 1$, then*

$$\left|f(a)f(b) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} f\left(\frac{ab}{x}\right)dx\right| \le \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}} H_1\left(1,q;h_1(\theta),h_1(\vartheta)\right),$$

$$\left|f^2\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} f\left(\frac{ab}{x}\right)dx\right| \le \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}} H_2\left(1,q;h_2(\theta),h_2(\vartheta)\right),$$

If taking $q = 1$ in Theorem 1, we can derive the following corollary.

Corollary 3. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, and $a, b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$. If $|f'|$ is geometrically convex on $[a,b]$ for $s \in (0,1]$, then*

$$\left|f(a)f(b) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} f\left(\frac{ab}{x}\right)dx\right| \le \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right) H_1\left(s,1;h_1(\theta),h_1(\vartheta)\right),$$

$$\left|f^2\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} f\left(\frac{ab}{x}\right)dx\right| \le \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right) H_2\left(s,1;h_2(\theta),h_2(\vartheta)\right),$$

Theorem 2. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, and $a, b \in I^\circ$ with $a < b$ and $f' \in L[a, b]$. If $|f'|^q$ is $s-$geometrically convex on $[a, b]$ for $q \geq 1$ and $s \in (0, 1]$, then*

(2.19)
$$\left| f(a)f(b) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} f\left(\frac{ab}{x}\right) dx \right| \leq \frac{1}{2} \ln\left(\frac{b}{a}\right) \left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} H_3(s, q; h_3(\theta), h_3(\vartheta)),$$

(2.20)
$$\left| f^2\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} f\left(\frac{ab}{x}\right) dx \right| \leq \frac{1}{2} \ln\left(\frac{b}{a}\right) \left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} H_3(s, q; h_3(\theta), h_3(\vartheta)),$$

where $M_1 = \max\limits_{x \in [a, \sqrt{ab}]} |f(x)|$, $M_2 = \max\limits_{x \in [\sqrt{ab}, b]} |f(x)|$,

$$h_3(\theta) = \begin{cases} 1, & \theta = 1 \\ \frac{\theta - 1}{\ln \theta}, & \theta \neq 1 \end{cases},$$

$$H_3(s, q; h_3(\theta), h_3(\vartheta))$$
$$= \begin{cases} b|f'(b)|^s M_1 h_3^{1/q}\left(\theta\left(\frac{qs}{2}, \frac{qs}{2}\right)\right) + a|f'(a)|^s M_2 h_3^{1/q}\left(\vartheta\left(\frac{qs}{2}, \frac{qs}{2}\right)\right), \\ \qquad |f'(a)|,\ |f'(b)| \leq 1, \\ b|f'(b)|^{1/s} M_1 h_3^{1/q}\left(\theta\left(\frac{q}{2s}, \frac{q}{2s}\right)\right) + a|f'(a)|^{1/s} M_2 h_3^{1/q}\left(\vartheta\left(\frac{q}{2s}, \frac{q}{2s}\right)\right), \\ \qquad |f'(a)|,\ |f'(b)| \geq 1, \\ b|f'(b)|^{1/s} M_1 h_3^{1/q}\left(\theta\left(\frac{qs}{2}, \frac{q}{2s}\right)\right) + a|f'(a)|^s M_2 h_3^{1/q}\left(\vartheta\left(\frac{qs}{2}, \frac{q}{2s}\right)\right), \\ \qquad |f'(a)| \leq 1 \leq |f'(b)|, \\ b|f'(b)|^s M_1 h_3^{1/q}\left(\theta\left(\frac{q}{2s}, \frac{qs}{2}\right)\right) + a|f'(a)|^{1/s} M_2 h_3^{1/q}\left(\vartheta\left(\frac{q}{2s}, \frac{qs}{2}\right)\right), \\ \qquad |f'(b)| \leq 1 \leq |f'(a)|. \end{cases},$$

and $\theta(u, v)$, $\vartheta(u, v)$ are the same as in (2.6).

Proof. (1) Let $M_1 = \max\limits_{x \in [a, \sqrt{ab}]} |f(x)|$, $M_2 = \max\limits_{x \in [\sqrt{ab}, b]} |f(x)|$. Since $|f'|^q$ is $s-$geometrically convex on $[a, b]$, from lemma 1 and Hölder inequality, we have

$$\begin{aligned} &\left| f(a)f(b) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} f\left(\frac{ab}{x}\right) dx \right| \\ \leq\ & \int_0^1 \frac{b}{2} \left|\ln\left(\frac{a}{b}\right)\right| |t-1| \left(\frac{a}{b}\right)^{\frac{t}{2}} \left| f\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right| \left| f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right| \\ &+ \frac{a}{2} \ln\left(\frac{b}{a}\right) |t-1| \left(\frac{b}{a}\right)^{\frac{t}{2}} \left| f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right| \left| f\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right| dt \\ \leq\ & \frac{b}{2} \left|\ln\left(\frac{a}{b}\right)\right| M_1 \int_0^1 (1-t) \left(\frac{a}{b}\right)^{\frac{t}{2}} \left| f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right| dt \\ &+ \frac{a}{2} \ln\left(\frac{b}{a}\right) M_2 \int_0^1 (1-t) \left(\frac{b}{a}\right)^{\frac{t}{2}} \left| f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right| dt \end{aligned}$$

$$\leq \frac{b}{2}\left|\ln\left(\frac{a}{b}\right)\right| M_1\left(\int_0^1 (1-t)^{\frac{q}{q-1}}\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(\frac{a}{b}\right)^{\frac{qt}{2}}\left|f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{a}{2}\ln\left(\frac{b}{a}\right) M_2\left(\int_0^1 (1-t)^{\frac{q}{q-1}}\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(\frac{b}{a}\right)^{\frac{qt}{2}}\left|f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{b}{2}\ln\left(\frac{b}{a}\right) M_1\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(\frac{a}{b}\right)^{\frac{qt}{2}}\left|f'(a)\right|^{q(t/2)^s}\left|f'(b)\right|^{q((2-t)/2)^s} dt\right)^{\frac{1}{q}}$$

$$+\frac{a}{2}\ln\left(\frac{b}{a}\right) M_2\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(\frac{b}{a}\right)^{\frac{qt}{2}}\left|f'(b)\right|^{q(t/2)^s}\left|f'(a)\right|^{q((2-t)/2)^s} dt\right)^{\frac{1}{q}}. \tag{2.21}$$

(i) If $1 \geq |f'(a)|,\ |f'(b)|$, by (2.9) we have

$$\int_0^1 \left(\frac{a}{b}\right)^{\frac{qt}{2}}\left|f'(a)\right|^{q(t/2)^s}\left|f'(b)\right|^{q((2-t)/2)^s} dt = \left|f'(b)\right|^{qs} h_3\left(\theta\left(\frac{qs}{2},\frac{qs}{2}\right)\right),$$

$$\int_0^1 \left(\frac{b}{a}\right)^{\frac{qt}{2}}\left|f'(b)\right|^{q(t/2)^s}\left|f'(a)\right|^{q((2-t)/2)^s} dt = \left|f'(a)\right|^{qs} h_3\left(\vartheta\left(\frac{qs}{2},\frac{qs}{2}\right)\right). \tag{2.22}$$

(ii) If $1 \leq |f'(a)|,\ |f'(b)|$, by (2.9) we have

$$\int_0^1 \left(\frac{a}{b}\right)^{\frac{qt}{2}}\left|f'(a)\right|^{q(t/2)^s}\left|f'(b)\right|^{q((2-t)/2)^s} dt \leq \left|f'(b)\right|^{q/s} h_3\left(\theta\left(\frac{q}{2s},\frac{q}{2s}\right)\right),$$

$$\int_0^1 \left(\frac{b}{a}\right)^{\frac{qt}{2}}\left|f'(b)\right|^{q(t/2)^s}\left|f'(a)\right|^{q((2-t)/2)^s} dt \leq \left|f'(a)\right|^{q/s} h_3\left(\vartheta\left(\frac{q}{2s},\frac{q}{2s}\right)\right). \tag{2.23}$$

(iii) If $|f'(a)| \leq 1 \leq |f'(b)|$, by (2.9) we obtain that

$$\int_0^1 \left(\frac{a}{b}\right)^{\frac{qt}{2}}\left|f'(a)\right|^{q(t/2)^s}\left|f'(b)\right|^{q((2-t)/2)^s} dt \leq \left|f'(b)\right|^{q/s} h_3\left(\theta\left(\frac{qs}{2},\frac{q}{2s}\right)\right),$$

$$\int_0^1 \left(\frac{b}{a}\right)^{\frac{qt}{2}}\left|f'(b)\right|^{q(t/2)^s}\left|f'(a)\right|^{q((2-t)/2)^s} dt \leq \left|f'(a)\right|^{qs} h_3\left(\vartheta\left(\frac{qs}{2},\frac{q}{2s}\right)\right). \tag{2.24}$$

(iv) If $|f'(b)| \leq 1 \leq |f'(a)|$, by (2.9) we obtain that

$$\int_0^1 \left(\frac{a}{b}\right)^{\frac{qt}{2}}\left|f'(a)\right|^{q(t/2)^s}\left|f'(b)\right|^{q((2-t)/2)^s} dt \leq \left|f'(b)\right|^{qs} h_3\left(\theta\left(\frac{q}{2s},\frac{qs}{2}\right)\right),$$

$$\int_0^1 \left(\frac{b}{a}\right)^{\frac{qt}{2}} \left|f'(b)\right|^{q(t/2)^s} \left|f'(a)\right|^{q((2-t)/2)^s} dt \le \left|f'(a)\right|^{q/s} h_3\left(\vartheta\left(\frac{q}{2s},\frac{qs}{2}\right)\right). \tag{2.25}$$

From (2.21) to (2.25), (2.19) holds.

(2) Let $M_1 = \max_{x\in[a,\sqrt{ab}]} |f(x)|$, $M_2 = \max_{x\in[\sqrt{ab},b]} |f(x)|$. Since $|f'|^q$ is $s-$geometrically convex on $[a,b]$, from lemma 1 and Hölder inequality, we have

$$\left|f^2\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} f\left(\frac{ab}{x}\right) dx\right|$$

$$\le \frac{b}{2}\ln\left(\frac{b}{a}\right) M_1 \left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \left(\int_0^1 \left(\frac{a}{b}\right)^{\frac{qt}{2}} \left|f'(a)\right|^{q(t/2)^s} \left|f'(b)\right|^{q((2-t)/2)^s} dt\right)^{\frac{1}{q}}$$

$$+\frac{a}{2}\ln\left(\frac{b}{a}\right) M_2 \left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \left(\int_0^1 \left(\frac{b}{a}\right)^{\frac{qt}{2}} \left|f'(b)\right|^{q(t/2)^s} \left|f'(a)\right|^{q((2-t)/2)^s} dt\right)^{\frac{1}{q}}. \tag{2.26}$$

From (2.26) and (2.22) to (2.25), (2.20) holds. □

If taking $s = 1$ in Theorem 2, we can derive the following corollary.

Corollary 4. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, and $a, b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is geometrically convex on $[a,b]$ for $q \ge 1$, then*

$$\left|f(a)f(b) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} f\left(\frac{ab}{x}\right) dx\right| \le \frac{1}{2}\ln\left(\frac{b}{a}\right)\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} H_3\left(1,q;h_3(\theta),h_3(\vartheta)\right),$$

$$\left|f^2\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} f\left(\frac{ab}{x}\right) dx\right| \le \frac{1}{2}\ln\left(\frac{b}{a}\right)\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} H_3\left(1,q;h_3(\theta),h_3(\vartheta)\right),$$

where $\theta(u,v)$, $\vartheta(u,v)$, $H_3(1,q;h_3(\theta),h_3(\vartheta))$ and $h_3(\theta)$ are the same as in Theorem 2.

3. Application to Special Means

Let us recall the following special means of two nonnegative number a, b with $b > a$:

(1) The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}.$$

(2) The Logarithmic mean

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}.$$

(3) The p-Logarithmic mean

$$L_p = L_p(a,b) := \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, \quad p \in \mathbb{R}\backslash\{-1,0\}.$$

Let $f(x)=(x^s/s)+1,\ x\in(0,1],\ 0<s<1,\ q\geq 1$, and then the function $|f'(x)|^q=x^{(s-1)q}$ is $s-$geometrically convex on $(0,1]$ for $0<s<1$ (see [2]).

Proposition 1. *Let $0<a<b\leq 1,\ 0<s<1$, and $q\geq 1$. Then for $s\neq\frac{1}{2}$*

$$\begin{aligned}&\left|G^2\left(\frac{a^s}{s}+1,\frac{b^s}{s}+1\right)-\frac{2}{s^2}A\left(G^2\left(a^s,b^s\right),s^2\right)-\frac{2}{s}L_{s-1}^{s-1}\left(a,b\right)L\left(a,b\right)\right|\\ \leq\ &\frac{1}{2}\left(\frac{b-a}{2L\left(a,b\right)}\right)^{1-\frac{1}{q}}\left[b^{1-\frac{1}{2s}}M_1\left\{\left(\frac{2s}{\left(2s-1\right)q}\right)\left[b^{q-\frac{q}{2s}}-L_{q-\frac{q}{2s}-1}^{q-\frac{q}{2s}-1}(a,b)L(a,b)\right]\right\}^{\frac{1}{q}}\right.\\ &\left.a^{1-\frac{1}{2s}}M_2\left\{\left(\frac{2s}{\left(2s-1\right)q}\right)\left[L_{q-\frac{q}{2s}-1}^{q-\frac{q}{2s}-1}(a,b)L(a,b)-a^{q-\frac{q}{2s}}\right]\right\}^{\frac{1}{q}}\right],\end{aligned}$$

$$\begin{aligned}&\left|\left(\frac{G^s\left(a,b\right)}{s}+1\right)^2-\frac{2}{s^s}A\left(G^2\left(a^s,b^s\right),s^2\right)-\frac{2}{s}L_{s-1}^{s-1}\left(a,b\right)L\left(a,b\right)\right|\\ \leq\ &\frac{1}{2}\left(\frac{b-a}{2L\left(a,b\right)}\right)^{1-\frac{1}{q}}\left[b^{1-\frac{1}{2s}}M_1\left\{\left(\frac{2s}{\left(2s-1\right)q}\right)\left[L_{q-\frac{q}{2s}-1}^{q-\frac{q}{2s}-1}(a,b)L(a,b)-a^{q-\frac{q}{2s}}\right]\right\}^{\frac{1}{q}}\right.\\ &\left.a^{1-\frac{1}{2s}}M_2\left\{\left(\frac{2s}{\left(2s-1\right)q}\right)\left[b^{q-\frac{q}{2s}}-L_{q-\frac{q}{2s}-1}^{q-\frac{q}{2s}-1}(a,b)L(a,b)\right]\right\}^{\frac{1}{q}}\right],\end{aligned}$$

for $s=\frac{1}{2}$

$$\begin{aligned}&\left|G^2\left(2\sqrt{a}+1,2\sqrt{b}+1\right)-8A\left(G^2\left(\sqrt{a},\sqrt{b}\right),\frac{1}{4}\right)-4L_{-1/2}^{-1/2}\left(a,b\right)L\left(a,b\right)\right|\\ \leq\ &\frac{1}{2}\left(\frac{b-a}{L\left(a,b\right)}\right)\left(\sqrt[4]{ab}+\sqrt{b}+1\right),\end{aligned}$$

$$\begin{aligned}&\left|\left(2\sqrt{G\left(a,b\right)}+1\right)^2-8A\left(G^2\left(\sqrt{a},\sqrt{b}\right),\frac{1}{4}\right)-4L_{-1/2}^{-1/2}\left(a,b\right)L\left(a,b\right)\right|\\ \leq\ &\frac{1}{2}\left(\frac{b-a}{L\left(a,b\right)}\right)\left(\sqrt[4]{ab}+\sqrt{b}+1\right).\end{aligned}$$

Proof. Let $f(x)=(x^s/s)+1,\ x\in(0,1],\ 0<s<1$. Then $|f'(a)|=a^{s-1}>b^{s-1}=|f'(b)|\geq 1$, $M_1=\max\limits_{x\in\left[a,\sqrt{ab}\right]}|f(x)|=\left(\sqrt{ab}^s/s\right)+1$, $M_2=\max\limits_{x\in\left[\sqrt{ab},b\right]}|f(x)|=(b^s/s)+1$, and for $s\neq\frac{1}{2}$

$$\begin{aligned}&b\left|f'(b)\right|^{1/s}M_1h_1^{1/q}\left(\theta\left(\frac{q}{2s},\frac{q}{2s}\right)\right)\\ =\ &\left(\frac{L(a,b)}{(b-a)}\right)^{\frac{1}{q}}b^{1-\frac{1}{2s}}M_1\left\{\left(\frac{2s}{\left(2s-1\right)q}\right)\left[b^{q-\frac{q}{2s}}-L_{q-\frac{q}{2s}-1}^{q-\frac{q}{2s}-1}(a,b)L(a,b)\right]\right\}^{\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}&a\left|f'(a)\right|^{1/s}M_2h_1^{1/q}\left(\vartheta\left(\frac{q}{2s},\frac{q}{2s}\right)\right)\\ =\ &\left(\frac{L(a,b)}{(b-a)}\right)^{\frac{1}{q}}a^{1-\frac{1}{2s}}M_2\left\{\left(\frac{2s}{\left(2s-1\right)q}\right)\left[L_{q-\frac{q}{2s}-1}^{q-\frac{q}{2s}-1}(a,b)L(a,b)-a^{q-\frac{q}{2s}}\right]\right\}^{\frac{1}{q}}\end{aligned}$$

$$b\left|f'(b)\right|^{1/s} M_1 h_2^{1/q}\left(\theta\left(\frac{q}{2s},\frac{q}{2s}\right)\right)$$
$$= \; = \left(\frac{L(a,b)}{(b-a)}\right)^{\frac{1}{q}} b^{1-\frac{1}{2s}} M_1 \left\{\left(\frac{2s}{(2s-1)\,q}\right)\left[L_{q-\frac{q}{2s}-1}^{q-\frac{q}{2s}-1}(a,b)L(a,b)-a^{q-\frac{q}{2s}}\right]\right\}^{\frac{1}{q}}$$

$$a\left|f'(a)\right|^{1/s} M_2 h_2^{1/q}\left(\vartheta\left(\frac{q}{2s},\frac{q}{2s}\right)\right)$$
$$= \; = \left(\frac{L(a,b)}{(b-a)}\right)^{\frac{1}{q}} a^{1-\frac{1}{2s}} M_2 \left\{\left(\frac{2s}{(2s-1)\,q}\right)\left[b^{q-\frac{q}{2s}}-L_{q-\frac{q}{2s}-1}^{q-\frac{q}{2s}-1}(a,b)L(a,b)\right]\right\}^{\frac{1}{q}}$$

for $s=\frac{1}{2}$

$$h_1\left(\theta\left(\frac{q}{2s},\frac{q}{2s}\right)\right)=h_2\left(\theta\left(\frac{q}{2s},\frac{q}{2s}\right)\right)=\frac{1}{2}.$$

By Theorem 1, Proposition 1 is thus proved. □

Proposition 2. *Let $0<a<b\leq 1,\ 0<s<1$, and $q>1$. Then for $s\neq\frac{1}{2}$*

$$\left|G^2\left(\frac{a^s}{s}+1,\frac{b^s}{s}+1\right)-\frac{2}{s^2}A\left(G^2\left(a^s,b^s\right),s^2\right)-\frac{2}{s}L_{s-1}^{s-1}\left(a,b\right)L\left(a,b\right)\right|$$
$$\leq \frac{b-a}{L(a,b)}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} A\left(a^{1-\frac{1}{s}},b^{1-\frac{1}{s}}\right)\left(L_{q-\frac{q}{2s}-1}^{q-\frac{q}{2s}-1}(a,b)L(a,b)\right)^{\frac{1}{q}},$$

$$\left|\left(\frac{G^s\left(a,b\right)}{s}+1\right)^2-\frac{2}{s^s}A\left(G^2\left(a^s,b^s\right),s^2\right)-\frac{2}{s}L_{s-1}^{s-1}\left(a,b\right)L\left(a,b\right)\right|$$
$$\leq \frac{b-a}{L(a,b)}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} A\left(a^{1-\frac{1}{s}},b^{1-\frac{1}{s}}\right)\left(L_{q-\frac{q}{2s}-1}^{q-\frac{q}{2s}-1}(a,b)L(a,b)\right)^{\frac{1}{q}},$$

and for $s=\frac{1}{2}$ we have

$$\left|G^2\left(2\sqrt{a}+1,2\sqrt{b}+1\right)-8A\left(G^2\left(\sqrt{a},\sqrt{b}\right),\frac{1}{4}\right)-4L_{-1/2}^{-1/2}\left(a,b\right)L\left(a,b\right)\right|$$
$$\leq \frac{b-a}{L\left(a,b\right)}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\sqrt[4]{ab}+\sqrt{b}+1\right),$$

$$\left|\left(2\sqrt{G\left(a,b\right)}+1\right)^2-8A\left(G^2\left(\sqrt{a},\sqrt{b}\right),\frac{1}{4}\right)-4L_{-1/2}^{-1/2}\left(a,b\right)L\left(a,b\right)\right|$$
$$\frac{b-a}{L\left(a,b\right)}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\sqrt[4]{ab}+\sqrt{b}+1\right).$$

Proof. Let $f(x)=(x^s/s)+1,\ x\in(0,1],\ 0<s<1$. Then $|f'(a)|=a^{s-1}>b^{s-1}=|f'(b)|\geq 1$, $M_1=\max_{x\in\left[a,\sqrt{ab}\right]}|f(x)|=\left(\sqrt{ab}^{\,s}/s\right)+1$, $M_2=\max_{x\in\left[\sqrt{ab},b\right]}|f(x)|=(b^s/s)+1$, and for $s\neq\frac{1}{2}$

$$h_3\left(\theta\left(\frac{q}{2s},\frac{q}{2s}\right)\right) = b^{\frac{q}{2s}-q}L_{q-\frac{q}{2s}-1}^{q-\frac{q}{2s}-1}(a,b)L(a,b),$$
$$h_3\left(\vartheta\left(\frac{q}{2s},\frac{q}{2s}\right)\right) = a^{\frac{q}{2s}-q}L_{q-\frac{q}{2s}-1}^{q-\frac{q}{2s}-1}(a,b)L(a,b),$$

for $s=\frac{1}{2}$ we have

$$h_3\left(\theta\left(\frac{q}{2s},\frac{q}{2s}\right)\right)=h_3\left(\vartheta\left(\frac{q}{2s},\frac{q}{2s}\right)\right)=1.$$

Using Theorem 2, Proposition 2 is thus proved. □

References

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